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Non-universal corrections to scaling for a lattice model

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Abstract. A field theoretic model derived from a lattice spin model is studied within the framework of the renormalisation group. To second order in $\varepsilon = 4 - d$, where d is the dimensionality of the lattice, the non-universal fixed point and the amplitude of the leading correction to scaling for the susceptibility are determined for two types of lattice. The functional form obtained for the amplitude compares qualitatively well with high-temperature series work; in particular, it gives a prediction for the amplitude of corrections to scaling for the spin-$\frac{1}{2}$ Ising model on a body centred cubic lattice in three dimensions that is in fair numerical agreement with high-temperature series estimates.

1. Introduction

Until some recent work had been performed (Nickel 1982, Roskies 1981) there existed a discrepancy between estimates for critical exponents which are expected to be universal on the basis of renormalisation group (RG) arguments. The values yielded by high-temperature series analysis (see Domb 1974, Gaunt 1982 for reviews) and those given by RG techniques (Baker *et al* 1976, 1978, Le Guillou and Zinn-Justin 1977, Zinn-Justin 1982) were not reconcilable within the estimated accuracies of the different methods. This difference was particularly marked for the exponent γ governing the divergence of the susceptibility as the critical temperature is approached. Nickel (1982) and Roskies (1981) introduced terms representing confluent (weaker) singularities into the high-temperature series analysis and studied the effect of these terms on the estimates for exponents. They found that the modified high-temperature series estimates could then be reconciled with the existing RG results. Using the value of the confluent singularity exponent calculated by RG methods, Nickel (1982) examined the general spin-s Ising model with nearest-neighbour interactions only on a body centred cubic lattice in three dimensions. He found that the amplitude of the confluent singularity term was a function of the spin s and in addition he was able to estimate the amplitude of the confluent singularity for the spin-$\frac{1}{2}$ model. Roskies (1981) also estimates the amplitude for the spin-$\frac{1}{2}$ model.

Confluent singularities or corrections to scaling arise naturally in the RG approach (Wegner 1972). It further predicts that the amplitude of the leading corrections to scaling should be a function of the coupling constant g, that it should be characterised by an exponent ω and that the amplitude vanishes if the coupling constant coincides with the fixed point $g = g^*$. The purpose of this paper is to study the field theory arising from the transformation of a discrete lattice spin model with nearest-neighbour interactions only. We seek to find the extent to which the results of Nickel's calculations are in agreement with field theoretic estimates for such non-universal quantities.

The lattice spin model we will use is defined in § 2. A field theory is obtained by performing a Hubbard transformation (Hubbard 1972, Baker 1962) on the lattice spin model. The field theory we obtain differs in a few respects from a normal $\phi^4(x)$ field theory. The presence of the lattice means that integrals in momentum space run over the Brillouin zone of the specific lattice type. In addition, the Hubbard transformation generates interactions of arbitrary order in ϕ^2 leading to a number of extra diagrams to be considered.

In § 3 we use the formulation of the renormalisation group given by Zinn-Justin (1973), that of working with the *bare* vertex functions to maintain contact with the parameters of the lattice Hamiltonian. The vertex functions are expressed as functions of the reduced temperature.

Section 4 contains the graphical expansion for the bare vertex functions. We keep graphs with up to two loops in order to calculate the RG functions $\beta(g)$, $\gamma_{\phi^2}(g)$, $\gamma_\phi(g)$ to second order in g and ε. We express these functions in terms of graphs to identify which graphs contribute at a given order in ε and g.

In order to reduce the number of graphs that must be evaluated we use arguments based on universality of exponents in § 5. The universality of exponents (the foremost result of RG studies) which are obtained from the RG functions $\beta(g)$, $\gamma_{\phi^2}(g)$, $\gamma_\phi(g)$ at the fixed point $g=g^*$ gives rise to conditions on the coefficients of these functions as power series in ε and g. This policy of using universal results greatly reduces the number of graphs that must be evaluated in general dimension d to provide an expansion in ε for non-universal quantities such as the fixed point and the amplitude of the leading corrections to scaling. The details of the transform methods that make possible the analytic continuation in d for the required graphs are given in appendices 1 and 2 for two different lattice types. The results enable the determination of the fixed point, g^*, to be made to $O(\varepsilon^2)$ in an asymptotic series for both the simple cubic (SC) and body centred cubic (BCC) lattices.

In § 6 an expression for the susceptibility above the critical temperature is derived. It is more complicated than in a $\phi^4(x)$ theory as the external field conjugate to the spin variable couples in a more complex way to the field ϕ_i.

We follow the prescription of Bruce and Wallace (1976) and solve the RG equations by the method of characteristics in § 7. By relating vertex functions near criticality to those far from criticality where the perturbation expansion may be trusted, the crossover scaling functional form of the susceptibility may be determined. This allows the functional form of the amplitude of the leading corrections to scaling to be found for the coupling constant g close to its fixed point value g^* and hence we can estimate the numerical value of the amplitude for the spin-$\frac{1}{2}$ Ising model on the SC and BCC lattices in three dimensions.

Section 8 summarises the results of our analysis. These are discussed and compared with estimates available from high-temperature series.

Appendices 1 and 2 deal with the analytic continuation in d for various integrals for the SC and BCC lattices respectively. This is done by means of integral transform methods.

Appendix 3 deals with the generalisation of the model described in § 2 to the classical Heisenberg case where the spin variable has n components. Similar analysis for this case is presented and again compared with the more sparse information available from high-temperature series.

2. Model

The model studied in this paper was proposed by Wallace (unpublished) and differs from the standard lattice spin model. It is formulated on a lattice of dimensionality d with lattice spacing Λ^{-1} and N sites. The dimensionless configurational energy has the form

$$-\frac{\mathscr{H}}{kT}=\frac{1}{2}\sum_{i,j=1}^{N} K_{ij}s_is_j+\sum_{i=1}^{N} H_is_i \tag{2.1}$$

with sums running over lattice sites i and j. The exchange coupling constant K_{ij} is a matrix connecting spins s_i and s_j at sites i and j respectively and we will consider the case of nearest-neighbour interactions only. The spatially dependent external field H_i couples linearly to the spin variable s_i. The spin variable s_i at each site i is the sum of L independent copies of a 'spin $\frac{1}{2}$' at that site normalised with a factor of $L^{-1/2}$.

$$s_i=L^{-1/2}\sum_{m=1}^{L}\sigma_{im} \qquad \text{with } \sigma_{im}=\pm 1;\ m=1,\ldots,L. \tag{2.2}$$

Two special cases of this model are of note. The case $L=1$ is the ordinary spin-$\frac{1}{2}$ Ising model. The case $L=\infty$ is the free Gaussian model.

In order to extract a field theory from this lattice model a Hubbard transformation (Hubbard 1972, Baker 1962) is used. This employs the trick of regarding the partition function, Z, as the result of performing a Gaussian integral with respect to a discrete field variable by completing the square in the field variable. This decouples the terms in the spin variables s_i and s_j so that the configurational sum over the spin variables in the partition function may be explicitly performed.

$$\begin{aligned} Z\{H_i\} &= \sum_{\{s_i\}} \exp(-\mathscr{H}/kT) \\ &= \sum_{\{s_i\}} C\int_{-\infty}^{\infty}\prod_{i=1}^{N} \mathrm{d}\phi_i \exp\left(-\tfrac{1}{2}\sum_{i,j=1}^{N} K_{ij}^{-1}\phi_i\phi_j+\sum_{i=1}^{N} s_i(\phi_i+H_i)\right) \\ &= C\int_{-\infty}^{\infty}\prod_{i=1}^{N} \mathrm{d}\phi_i \exp\left(-\tfrac{1}{2}\sum_{i,j=1}^{N} K_{ij}^{-1}\phi_i\phi_j+\sum_{i=1}^{N} L\ln\cosh[(\phi_i+H_i)L^{-1/2}]\right) \end{aligned} \tag{2.3}$$

where C is a constant independent of ϕ_i and H_i which contributes only to an overall scale factor and will be omitted in the following.

Interaction terms in ϕ_i will be generated by expanding the ln cosh term of (2.3) for zero external field. One of these terms will be quadratic in ϕ_i and it is convenient to separate the terms occurring in the exponential in (2.3) into two pieces,

$$-\tfrac{1}{2}\sum_{i,j=1}^{N}\phi_i\phi_jK_{ij}^{-1}+\sum_{i=1}^{N} L\ln\cosh(\phi_iL^{-1/2})=\mathscr{H}^{(2)}+\mathscr{H}_{\text{int}} \tag{2.4}$$

with

$$\mathscr{H}^{(2)}=-\tfrac{1}{2}\sum_{i,j=1}^{N}\phi_i\phi_jK_{ij}^{-1}+\tfrac{1}{2}\sum_{i=1}^{N}\phi_i^2, \tag{2.5}$$

$$\mathscr{H}_{\text{int}}=\sum_{i=1}^{N}[L\ln\cosh(\phi_iL^{-1/2})-\tfrac{1}{2}\phi_i^2]. \tag{2.6}$$

The reduced Hamiltonian $\mathscr{H}^{(2)}$ defined in (2.5) contains all the terms quadratic in the field while $\mathscr{H}_{\text{int}}$ defined in (2.6) specifies the interaction terms.

We introduce Fourier variables by

$$\phi_i = N^{-1} \sum_{k} \phi_k \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}_i) \tag{2.7}$$

where $\boldsymbol{r}_i$ is the position vector of the site i and the summation over $\boldsymbol{k}$ extends over the Brillouin zone of the lattice. For an isotropic nearest-neighbour exchange coupling constant K_{ij} we may immediately express (2.5) in momentum space using (2.7),

$$\mathscr{H}^{(2)} = -\tfrac{1}{2}N^{-1} \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}}\phi_{-\boldsymbol{k}}(K^{-1}(\boldsymbol{k}) - 1) \tag{2.8}$$

where

$$K(\boldsymbol{k}) = \sum_{(i-j)} K_{ij} \exp[\mathrm{i}\boldsymbol{k}(\boldsymbol{r}_i - \boldsymbol{r}_j)] = K \sum_{n=1}^{\nu} \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{a}_n) \tag{2.9}$$

where $\boldsymbol{a}$ are the lattice vectors of the nearest neighbours, K is the magnitude of the isotropic exchange coupling constant and ν is the coordination number of the lattice.

Using the relation (2.7) we may also express the interaction terms given by (2.6) in momentum space as follows:

$$\begin{aligned}\mathscr{H}_{\text{int}} = &-(12LN^4)^{-1} \sum_{k_1\dots k_4} \prod_{n=1}^{4} \phi_{k_n} \Delta\Big(\sum_{n=1}^{4} k_n\Big) + (45L^2N^6)^{-1} \sum_{k_1\dots k_6} \prod_{n=1}^{6} \phi_{k_n} \Delta\Big(\sum_{n=1}^{6} k_n\Big) \\ &-17(2520L^3N^8)^{-1} \sum_{k_1\dots k_8} \prod_{n=1}^{8} \phi_{k_n} \Delta\Big(\sum_{n=1}^{8} k_n\Big) + \mathrm{O}(L^{-4})\end{aligned} \tag{2.10}$$

where the vector nature of the momenta $\{k\}$ has been suppressed and the Δ function imposes momentum conservation at a vertex modulo a reciprocal lattice vector $\boldsymbol{G}$ by

$$\begin{aligned}\Delta\Big(\sum_{n=1}^{x} k_n\Big) &= N \qquad \text{if } \sum_{n=1}^{x} \boldsymbol{k}_n = \boldsymbol{G} \text{ for } \boldsymbol{G} \text{ some reciprocal lattice vector,} \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{2.11}$$

We consider the thermodynamic limit in which the number of sites of the lattice, N, tends to infinity. In this limit the discrete sum over the N points on the Brillouin zone is replaced by an integral over the momentum k according to the prescription for a general function $f(\phi_k)$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k} f(\phi_k) \to \frac{V}{(2\pi)^d} \int^{\text{BZ}} \mathrm{d}^d k\, f(\phi_k) \tag{2.12}$$

where the integral runs over the Brillouin zone of the lattice and V is the volume of a unit cell. For a lattice of general dimensionality d and a lattice spacing Λ^{-1}, V takes the form

$$V = w\Lambda^{-d} \tag{2.13}$$

where w is a dimensionless constant dependent on the lattice type but is independent of d for the SC and BCC lattices.

Using (2.12) for taking the thermodynamic limit we may write for (2.8)

$$\mathcal{H}^{(2)} = -\frac{V}{(2\pi)^d}\int^{\mathrm{BZ}} \mathrm{d}^d k\ \tfrac{1}{2}(K^{-1}(k)-1)\phi_k\phi_{-k}. \tag{2.14}$$

The field ϕ_k is now rescaled so that it assumes its usual canonical dimensions in terms of the momentum scale Λ. This is to highlight certain features of the calculation by making canonical dimensions explicit and is for convenience only:

$$\phi(k) = V^{1/2}\Lambda^{-1}\phi_k. \tag{2.15}$$

Expressing (2.14) in terms of the rescaled field $\phi(k)$ defined in (2.15) yields

$$\mathcal{H}^{(2)} = -(2\pi)^{-d}\int^{\mathrm{BZ}} \mathrm{d}^d k\ \tfrac{1}{2}(\Lambda^2 K^{-1}(k)-\Lambda^2)\phi(k)\phi(-k). \tag{2.16}$$

Equation (2.16) together with (2.9) for $K^{-1}(k)$ and the definition of the zeroth-order bare propagator in momentum space by

$$\langle\phi(q_1)\phi(q_2)\rangle = G_{\mathrm{b}}^{(2)}(q_1, K, \Lambda)\delta^{(d)}(q_1+q_2) \tag{2.17}$$

enables an explicit form for the propagator to be given,

$$\begin{aligned} G_{\mathrm{b}}^{(2)}(k, K, \Lambda) &= \left[\frac{\Lambda^2}{K}\left(\sum_{n=1}^{\nu} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_n)\right)^{-1} - \Lambda^2\right]^{-1} \\ &= \sum_{n=1}^{\nu} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_n)\Big/\left(\frac{\Lambda^2}{K} - \Lambda^2\sum_{n=1}^{\nu}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{a}_n)\right). \end{aligned} \tag{2.18}$$

We now examine the interaction terms of the theory given by (2.10) for $\mathcal{H}_{\mathrm{int}}$ in terms of the rescaled field $\phi(k)$ defined in (2.15) after taking the thermodynamic limit using (2.12). We may use the property that the propagator (2.18) is invariant under the replacement of momentum $\boldsymbol{q}$ by $\boldsymbol{q}+\boldsymbol{G}$ where $\boldsymbol{G}$ is some reciprocal lattice vector to express the Δ-function of (2.11) by a δ function in the following fashion. We may order the interaction terms as a power series in the dimensionless coupling constant g defined by

$$g = 2wL^{-1} \tag{2.19}$$

giving for (2.16)

$$\begin{aligned} \mathcal{H}_{\mathrm{int}} = &-\frac{g\Lambda^{\varepsilon}}{4!}\frac{1}{(2\pi)^{3d}}\prod_{i=1}^{3}\left(\int^{\mathrm{BZ}} \mathrm{d}^d\boldsymbol{q}_i\,\phi(\boldsymbol{q}_i)\right)\phi\left(-\sum_{i=1}^{3}\boldsymbol{q}_i\right) \\ &+\frac{4g^2\Lambda^{2\varepsilon-2}}{6!}\frac{1}{(2\pi)^{5d}}\prod_{i=1}^{5}\left(\int^{\mathrm{BZ}} \mathrm{d}^d\boldsymbol{q}_i\,\phi(\boldsymbol{q}_i)\right)\phi\left(-\sum_{i=1}^{5}\boldsymbol{q}_i\right) \\ &-\frac{34g^3\Lambda^{3\varepsilon-4}}{8!}\frac{1}{(2\pi)^{7d}}\prod_{i=1}^{7}\left(\int^{\mathrm{BZ}} \mathrm{d}^d\boldsymbol{q}_i\,\phi(\boldsymbol{q}_i)\right)\phi\left(-\sum_{i=1}^{7}\boldsymbol{q}_i\right) + \mathrm{O}(g^4). \end{aligned} \tag{2.20}$$

This expression keeps track of all Umklapp processes which arise from the original Δ-function. It should be noted that no approximation has been made in obtaining a field theory in terms of the continuous variable $\phi(k)$ other than taking the thermodynamic limit. We have a well defined field theory with a bare propagator in momentum space given by (2.18) which explicitly displays the presence of the lattice and interaction terms given by (2.20) where the canonical dimension of the vertex is made explicit (see Brézin 1982).

3. Renormalisation group

We examine the field theory by considering the N-point vertex functions. These vertex functions are the sum of all the connected one-particle-irreducible graphs with N external legs each carrying a momentum $\boldsymbol{q}_r$ $(r=1,\ldots,N)$ (N is now a parameter, not the number of sites). In the following all indices on the momenta will be suppressed. The vertex functions may be expanded as a power series in g and additionally depend on K, Λ and $\{q\}$.

The formulation of the renormalisation group we shall use is due to Zinn-Justin (1973) (Amit 1978). The RG equations determine the asymptotic behaviour of the bare vertex functions as criticality is approached. In this limit they become independent of the scale of the cut-off, Λ. The equation for the N-point vertex function takes the form

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\gamma_{\phi^2}(g)t\frac{\partial}{\partial t}-\frac{N}{2}\gamma_\phi(g)\right)\Gamma^{(N)}(q;t,g,\Lambda)=\Delta\Gamma^{(N)} \tag{3.1}$$

where t is a measure of the deviation of the temperature variable K from its true critical value K_c. The term $\Delta\Gamma^{(N)}$ is smaller by order t/Λ^2, q^2/Λ^2 up to powers of $\ln(t/\Lambda^2)$, $\ln(q^2/\Lambda^2)$ in an ε expansion compared with the leading terms retained in $\Gamma^{(N)}(q;t,g,\Lambda)$. The region $t\ll\Lambda^2$, $q^2\ll\Lambda^2$ is the critical regime we aim to study. Keeping only the leading terms in the vertex function in this critical regime leads to the homogeneous equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda}+\beta(g)\frac{\partial}{\partial g}+\gamma_{\phi^2}(g)t\frac{\partial}{\partial t}-\frac{N}{2}\gamma_\phi(g)\right)\Gamma^{(N)}(q;t,g,\Lambda)=0. \tag{3.2}$$

The RG functions $\beta(g)$, $\gamma_{\phi^2}(g)$, $\gamma_\phi(g)$ may be obtained as power series in g and ε by applying equation (3.2) to three independent vertex functions and solving the three equations simultaneously. Suitable choices for these independent vertex functions are $\Gamma^{(4)}(0;t,g,\Lambda)$, $\Gamma^{(2)}(0;t,g,\Lambda)$, $\Gamma^{(2)}(q;0,g,\Lambda)$.

First we must express the vertex functions as a function of the variable t, the reduced temperature, instead of K. The introduction of t can be accomplished by mass renormalisation using a counterterm. We start by identifying the zero of the two-point vertex function at zero external momentum calculated with the bare propagator as a function of K with the zero of the two-point vertex function calculated with a renormalised propagator and a mass counterterm,

$$\begin{aligned}&\Gamma^{(2)}(0;K=K_c,g,\Lambda)=0\\ &\equiv\Gamma^{(2)}(0;t=0,g,\Lambda)=0.\end{aligned} \tag{3.3}$$

A suitable definition of the counterterm is achieved by decomposing (2.16) for $\mathscr{H}^{(2)}$ in the following fashion:

$$\mathscr{H}^{(2)}=\frac{-1}{(2\pi)^d}\int^{\mathrm{BZ}}\mathrm{d}^dq\,\frac{1}{2}\bigg[\frac{\Lambda^2(K_c^{-1}-\nu)}{\nu}\phi(q)\phi(-q)+(\nu K_c)^{-1}\times\left(\frac{\Lambda^2K_c\nu/K-\Lambda^2\sum_{n=1}^{\nu}\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_n)}{\sum_{n=1}^{\nu}\exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_n)}\right)\phi(q)\phi(-q)\bigg]. \tag{3.4}$$

The first term in $\phi(q)\phi(-q)$ in (3.4) is momentum independent and is the counterterm. The second term in (3.4) in $\phi(q)\phi(-q)$ is the inverse of the renormalised propagator.

Condition (3.3) for the two-point vertex function requires that the mass counterterm is O(g) so that

$$K_c^{-1} = \nu + O(g). \tag{3.5}$$

This has immediate consequences for the renormalised propagator. The prefactor $(\nu K_c)^{-1}$ in (3.4) can be expressed as a power series in g with coefficients determined by (3.3) for the mass counterterm. There will be additional terms at each order in g in a graphical expansion coming from this expansion of the renormalised propagator in terms of g. For calculational purposes it is convenient to separate the power series in g arising from the factor $(\nu K_c)^{-1}$ and define an effective propagator

$$\hat{G}_0^{(2)}(q, t, \Lambda) = \sum_{n=1}^{\nu} \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_n) \bigg/ \left(\frac{\Lambda^2 K_c \nu}{K} - \Lambda^2 \sum_{n=1}^{\nu} \exp(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{a}_n)\right). \tag{3.6}$$

Integrals calculated with this effective propagator give rise to terms of the form $\ln t/\Lambda^2$ where the reduced temperature t is defined by

$$t = \Lambda^2(K_c/K - 1). \tag{3.7}$$

4. Graphical expansion

The graphs with up to two loops contributing to the two- and four-point vertex functions are displayed in figures 1 and 2. It is convenient to classify the graphs by their order in g including contributions arising from expanding the counterterm and prcpagators as power series in g. The overall sign of the graphical contribution will be made explicit, as will the canonical dimension in terms of the momentum scale of the graph:

$$\Gamma^{(2)}(q; t, g, \Lambda) = [\hat{G}_0^{(2)}(q, t, \Lambda)]^{-1} + g\Sigma_1(q, t) - g^2\Sigma_2(q, t) + O(g^3), \tag{4.1}$$

$$\Gamma^{(4)}(q; t, g, \Lambda) = \Lambda^{\varepsilon}(g - g^2\Pi_1(q, t) + g^3\Pi_2(q, t) + O(g^4)). \tag{4.2}$$

$\Sigma_1(q, t)$ is the sum of figures 1(b) and 1(c) plus a contribution from figure 1(a) at O(g). $\Sigma_2(q, t)$ is the sum of figures 1(d)–1(f) plus contributions from figures 1(a)–1(c). $\Pi_1(q, t)$ denotes figures 2(b) and 2(c) and $\Pi_2(q, t)$ is the sum of figures 2(d)–2(l) plus contributions from 2(b) and 2(c) at the appropriate order.

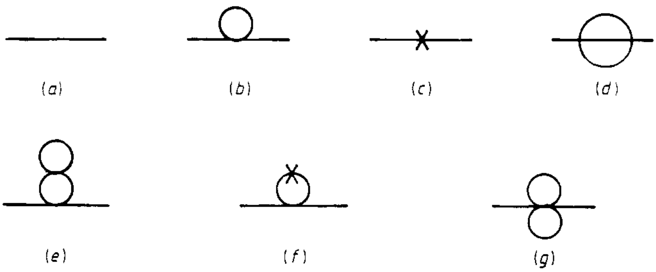

Figure 1. Graphs with up to two loops contributing to the two-point vertex function after mass renormalisation. The cross on the propagator denotes the mass counterterm.

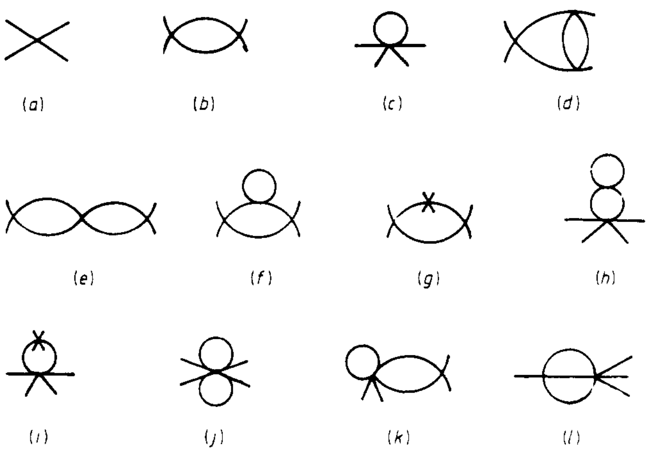

Figure 2. Graphs with up to two loops contributing to the four-point vertex function after mass renormalisation.

The presence of six- and eight-point vertices gives rise to additional diagrams such as figures 1(g), 2(c) and 2(k) not seen in a pure $\phi^4(x)$ theory. These diagrams have a higher degree of ultraviolet divergence, but this is tempered by dimensional factors present in the six- and eight-point coupling constants in (2.20). The result of this trade-off is that diagrams containing a six- or eight-point vertex are no more divergent than the normal $\phi^4(x)$ theory graphs (Brézin 1982).

We now apply the RG equation (3.2) to the vertex functions given in (4.1) and (4.2) for the three independent vertex functions $\Gamma^{(2)}(q; 0, g, \Lambda)$, $\Gamma^{(2)}(0; t, g, \Lambda)$ and $\Gamma^4(0; t, g, \Lambda)$. Solving the equations simultaneously allows the RG functions $\beta(g)$, $\gamma_{\phi^2}(g)$, $\gamma_\phi(g)$ to be expressed as the sum of diagrams. The only graphs dependent on the external momentum for the two-point functions are figures 1(a) and 1(d) which simplifies the computation of $\Sigma_2(q, 0)$:

$$\gamma_\phi(g) = -g^2[\hat{G}_0^{(2)}(q, 0, \Lambda)\Sigma_2(q, 0)|_{q=0}] + \mathrm{O}(g^3, \varepsilon g^2), \tag{4.3}$$

$$\begin{aligned} t\gamma_{\phi^2}(g) = {} & -g\Lambda\frac{\partial}{\partial\Lambda}\Sigma_1(0, t) + \varepsilon g\Sigma_1(0, t) \\ & + g^2\left[\Lambda\frac{\partial}{\partial\Lambda}\Sigma_2(0, t) + \Sigma_1(0, t)\Lambda\frac{\partial}{\partial\Lambda}(t^{-1}\Sigma_1(0, t) - \Pi_1(0, t))\right. \\ & \left. - \tfrac{1}{2}t^{-1}\left(\Lambda\frac{\partial}{\partial\Lambda}\Sigma_1(0, t)\right)^2 + t\gamma_\phi g^{-2}\right] + \mathrm{O}(\varepsilon g^2, g^3), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \beta(g) = {} & -\varepsilon g + g^2\Lambda\frac{\partial}{\partial\Lambda}\Pi_1(0, t) - \varepsilon g\Pi_1(0, t) \\ & - g^3\left[\Lambda\frac{\partial}{\partial\Lambda}\Pi_2(0, t) - 2\Pi_1(0, t)\Lambda\frac{\partial}{\partial\Lambda}\Pi_1(0, t) - \tfrac{1}{2}t^{-1}\left(\Lambda\frac{\partial}{\partial\Lambda}\Pi_1(0, t)\right)\right. \\ & \left. \times\left(\Lambda\frac{\partial}{\partial\Lambda}\Sigma_1(0, t)\right) - 2\gamma_\phi g^{-2}\right] + \mathrm{O}(g^2\varepsilon^2, g^3\varepsilon, g^4). \end{aligned} \tag{4.5}$$

These RG functions given by (4.3), (4.4) and (4.5) are finite in the limit $t/\Lambda^2 \to 0$ so any divergences in individual diagrams must cancel. We now use the idea of universality to simplify the calculation by telling us what diagrams must be evaluated in order to obtain these functions to second order.

5. Universality

Scaling behaviour of the vertex functions is obtained at the fixed point $g = g^*$ where $\beta(g^*) = 0$. The exponents obtained from $\gamma_{\phi^2}(g^*)$ and $\gamma_\phi(g^*)$ are universal, i.e. do not depend on the details of the Hamiltonian. The universality class of the Hamiltonian depends only on general features such as the dimensionality of the system and the number of components of the field. This universality requirement imposes conditions on the RG functions as follows. We expand these functions as power series in ε and g;

$$\beta(g) = -\varepsilon g + \beta_0 g^2 - \beta_1 \varepsilon g^2 - \beta_2 g^3 + O(g^4, \varepsilon g^3, \varepsilon^2 g^2), \tag{5.1}$$

$$\gamma_{\phi^2}(g) = \gamma_0 g + \gamma_1 \varepsilon g + \gamma_2 g^2 + O(g^3, \varepsilon g^2), \tag{5.2}$$

$$\gamma_\phi(g) = \gamma_3 g^2 + O(\varepsilon g^2, g^3). \tag{5.3}$$

Then the condition, using equation (5.1) for $\beta(g)$,

$$\beta(g^*) = 0$$

implies that the fixed point is given by

$$g^* = (\varepsilon/\beta_0)(1 + \varepsilon(\beta_2/\beta_0^2 + \beta_1/\beta_0) + O(\varepsilon^2)). \tag{5.4}$$

The universal exponent governing the corrections to scaling is given by

$$\omega \equiv \beta'(g^*) = \varepsilon - (\beta_2/\beta_0^2)\varepsilon^2 + O(\varepsilon^3). \tag{5.5}$$

The ratio β_2/β_0^2 is fixed by the universality class of the Hamiltonian. There is always the freedom to rescale the coupling constant g which is reflected in the numerical value of β_0. Having determined β_0, however, the value of β_2 is uniquely prescribed. The graphical content of β_2 can be found from (4.5). It should be noted that all contributions to β_2 arising from the presence of six- and eight-point vertices explicitly cancel, so that as required β_2 is independent of their inclusion in the field theoretic model. We have thus reduced the task of finding the non-universal fixed point to $O(\varepsilon^2)$ to finding the contributions to the terms β_0 and β_1.

Similarly the universality of the exponent η obtained from $\gamma_\phi(g^*)$ means that γ_3 is determined up to the scale of g and hence β_0. Universality of the exponent ν requires that $\gamma_{\phi^2}(g^*)$ has given coefficients as a power series in ε, again up to rescaling g:

$$\gamma_{\phi^2}(g^*) = \frac{\gamma_0}{\beta_0}\varepsilon\left[1 + \varepsilon\left(\frac{\beta_1}{\beta_0} + \frac{\beta_2}{\beta_0^2} + \frac{\gamma_1}{\gamma_0} + \frac{\gamma_2}{\gamma_0\beta_0}\right) + O(\varepsilon^2)\right]. \tag{5.6}$$

This allows us to express γ_2 in terms of β_1, β_2 and γ_1. This means that only the one-loop graphs that contribute to β_0, β_1 and γ_1 need be determined in order to perform a two-loop calculation for the amplitude of the leading correction to scaling in the susceptibility.

The only integrals that need to be considered are therefore the one-loop integrals with one and two powers of the propagator given respectively by

$$I_1(t, \Lambda) = (2\pi)^{-d} \int^{BZ} d^d q\, \hat{G}_0^{(2)}(q, t, \Lambda), \tag{5.7}$$

$$I_2(t, \Lambda) = (2\pi)^{-d} \int^{BZ} d^d q\, [\hat{G}_0^{(2)}(q, t, \Lambda)]^2. \tag{5.8}$$

After factoring off the explicit $\Lambda^{-\varepsilon}$ momentum dependence which will be cancelled by contributions from the coupling constants, we may expand the integrals as a power series in ε to obtain the general forms in the limit $t/\Lambda^2 \to \infty$:

$$I_1(t, \Lambda) = A_0\Lambda^2 - \tfrac{1}{2}B_1 t \ln(t/\Lambda^2) + C_1 t - \tfrac{1}{2}\varepsilon D_1 t \ln(t/\Lambda^2) + \tfrac{1}{4}\varepsilon E_1 t \ln^2(t/\Lambda^2) + \varepsilon F_1 t + \varepsilon K_0 \Lambda^2, \tag{5.9}$$

$$I_2(t, \Lambda) = -\tfrac{1}{2}B_2 \ln(t/\Lambda^2) + C_2 - \tfrac{1}{2}\varepsilon D_2 \ln(t/\Lambda^2) + \tfrac{1}{4}\varepsilon E_2 \ln^2(t/\Lambda^2) + \varepsilon F_2. \tag{5.10}$$

We now wish to express $\Pi_1(q, t)$, $\Sigma_1(q, t)$ in terms of these coefficients and so to find β_0, β_1, γ_0 and γ_1. Using the mass counterterm consistency equation (3.3), we may now evaluate the O(g) term in (3.5) that appears as a prefactor before $\hat{G}_0^{(2)}(q, t, \Lambda)$ in (3.4). This enables the contribution to $\Sigma_1(q, t)$ from figure 1(a) to be expressed in terms of the coefficients of (5.9), as well as obtaining the mass counterterm itself to the correct order:

$$\Sigma_1(0, t) = \tfrac{1}{2}[I_1(t, \Lambda) - I_1(0, \Lambda)] - \tfrac{1}{2}I_1(0, \Lambda)\Lambda^{-2}t, \tag{5.11}$$

$$\Pi_1(0, t) = \tfrac{3}{2}I_2(t, \Lambda) + 2\Lambda^{-2}I_1(t, \Lambda). \tag{5.12}$$

We now express equations (4.4) and (4.5) for the RG function $\beta(g)$ and $\gamma_{\phi^2}(g)$ in terms of the coefficients defined in (5.9) and (5.10) using (5.11) and (5.12). Comparing the resulting expressions with (5.1) and (5.2) order by order in ε and g allows the following identifications to be made:

$$\beta_0 = \tfrac{3}{2}B_2, \qquad \beta_1 = \tfrac{3}{2}(C_2 - D_2) + 2A_0, \qquad \gamma_0 = -\tfrac{1}{2}B_1, \qquad \gamma_1 = \tfrac{1}{2}(C_1 - D_1 - A_0), \tag{5.13}$$

together with the consistency conditions,

$$E_2 = \tfrac{1}{2}B_2, \qquad E_1 = \tfrac{1}{2}B_1, \tag{5.14}$$

which ensure that β_1 and γ_1 remain finite as $t/\Lambda^2 \to 0$.

By using the idea of universality we have thus reduced the problem to the task of determining the coefficients $B_2, C_2, D_2, B_1, C_1, D_1$ and A_0 of the asymptotic behaviour of the two integrals I_2 and I_1 in the regime $t/\Lambda^2 \to 0$.

These coefficients are determined for the SC and BCC lattices in appendices 1 and 2 respectively and the results are displayed in table 1.

6. Susceptibility

The susceptibility for the original lattice model in the high-temperature phase in a uniform external field H is given by

$$\chi = \partial M_i / \partial H|_{H=0} \tag{6.1}$$

Table 1.

	Simple cubic	Body centred cubic
A_0	0.239	0.237
B_1	−0.811	−0.811
C_1	−0.974	0.276
D_1	0.175	−0.230
B_2	0.811	0.811
C_2	0.058	−0.794
D_2	0.386	−0.051
g^*	$\frac{1}{12}\pi^2\varepsilon(1+0.618\varepsilon)$	$\frac{1}{12}\pi^2\varepsilon(1+0.103\varepsilon)$

where the magnetisation at the site i, M_i, is given by

$$M_i = \langle s_i \rangle = \partial \ln Z\{H\}/\partial H_i. \tag{6.2}$$

Then using equation (2.3) for $Z\{H\}$ we obtain for the susceptibility

$$\begin{aligned}\chi &= \frac{1}{N}\sum_{i,j=1}^{N} \frac{\partial}{\partial H_i}\frac{\partial}{\partial H_j}\ln Z\{H\} \\ &= \frac{1}{N}\sum_{i,j=1}^{N} L\langle \tanh(\phi_i L^{-1/2})\tanh(\phi_j L^{-1/2})\rangle + \frac{1}{N}\sum_{i=1}^{N}\langle \operatorname{sech}^2\phi_i\rangle. \end{aligned}\tag{6.3}$$

We may expand the hyperbolic function in (6.3) as a power series in L^{-1} whose coefficients are correlation functions in the fields ϕ_i and ϕ_j. The second term in (6.3) is purely local and is negligible compared with the first term in the thermodynamic limit and will be neglected in the following. We may examine this expression in terms of the rescaled momentum variables $\phi(q)$ defined in (2.15) from (2.7). The expansion in L^{-1} now becomes an expansion in g defined in (2.19) with the result for the susceptibility in terms of Green functions in momentum space:

$$\begin{aligned}\chi &= G^{(2)}(0; t, g, \Lambda) - \tfrac{1}{3}g\Lambda^{\varepsilon-2}G_3^{(1)}(0; t, g, \Lambda) + \tfrac{1}{15}g^2\Lambda^{4-2\varepsilon}G_5^{(1)}(0; t, g, \Lambda) \\ &\quad + \tfrac{1}{36}g^2\Lambda^{2\varepsilon-4}G_{3,3}^{(0)}(0; t, g, \Lambda) + O(g^3). \end{aligned}\tag{6.4}$$

In (6.4) a subscript on the Green functions denotes the insertion of one composite operator in the field $\phi'(q)$ of the degree given in the subscript. These insertions are made at zero momentum. The superscript refers to the number of external legs as before.

We decompose the Green functions with insertions of the composite operators in terms of vertex functions

$$\Lambda^{\varepsilon-2}G_3^{(1)}(0; t, g, \Lambda) = G^{(2)}(0; t, g, \Lambda)[\Lambda^{\varepsilon-2}\Gamma_3^{(1)}(0; t, g, \Lambda)], \tag{6.5}$$

$$\Lambda^{2\varepsilon-4}G_5^{(1)}(0; t, g, \Lambda) = G^{(2)}(0; t, g, \Lambda)[\Lambda^{\varepsilon-2}\Gamma_5^{(1)}(0; t, g, \Lambda)], \tag{6.6}$$

$$\begin{aligned}\Lambda^{2\varepsilon-4}G_{3,3}^{(0)}(0; t, g, \Lambda) &= G^{(2)}(0; t, g, \Lambda)[\Lambda^{\varepsilon-2}\Gamma_3^{(1)}(0; t, g, \Lambda)]^2 \\ &\quad + \Lambda^{2\varepsilon-4}\Gamma_{3,3}^{(0)}(0; t, g, \Lambda). \end{aligned}\tag{6.7}$$

We now examine the vertex functions occurring in (6.5), (6.6) and (6.7) as power series in g to the required order for substitution into (6.4). The graphs contributing to the vertex functions are given in figures 3, 4 and 5. Let figure 3(a) be denoted by

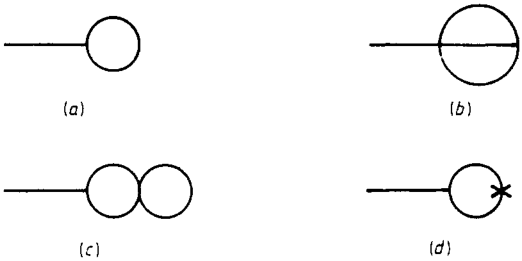

Figure 3. Graphs with up to two loops contributing to the one-point vertex function with the insertion of a composite three-point operator.

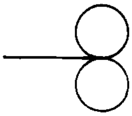

Figure 4. Graphs with up to two loops contributing to the one-point vertex function with the insertion of a composite five-point operator.

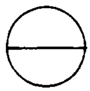

Figure 5. Graphs with up to two loops contributing to the zero-point vertex function with the insertion of two composite three-point operators.

$H_1(q, t)$, figures 3(b), 3(c) and 3(d) by $H_2(q, t)$, figure 4 by $H_3(q, t)$ and figure 5 by $H_4(q, t)$:

$$\Lambda^{\varepsilon-2}\Gamma_3^{(1)}(0, t, g, \Lambda) = \Lambda^{\varepsilon-2}[H_1(0, t) - gH_2(0, t) + \mathrm{O}(g^2)], \tag{6.8}$$

$$\Lambda^{2\varepsilon-4}\Gamma_5^{(1)}(0, t, g, \Lambda) = \Lambda^{2\varepsilon-4}[H_3(0, t) + \mathrm{O}(g)], \tag{6.9}$$

$$\Lambda^{2\varepsilon-4}\Gamma_{3,3}^{(0)}(0, t, g, \Lambda) = \Lambda^{2\varepsilon-4}[H_4(0, t) + \mathrm{O}(g)]. \tag{6.10}$$

The result of the explicit powers of the momentum scale multiplying the graphs is to pick out only the leading ultraviolet divergence of the graphs eliminating the dependence on t. In (6.8) and (6.9) the momentum factors cancel, leaving only a numerical coefficient. These may be written as

$$\Lambda^{\varepsilon-2}\Gamma_3^{(1)}(0, t, g, \Lambda) = 3A_0 - g\theta + \mathrm{O}(g^2) + \mathrm{O}(t/\Lambda^2), \tag{6.11}$$

$$\Lambda^{2\varepsilon-4}\Gamma_5^{(1)}(0; t, g, \Lambda) = 15A_0^2 + \mathrm{O}(g) + \mathrm{O}(t/\Lambda^2). \tag{6.12}$$

θ represents the coefficient of the leading ultraviolet divergence of the graphs in $H_2(0, t)$ and A_0 is defined in (4.9).

$H_4(0, t)$ has only $\Lambda^{2\varepsilon-2}$ ultraviolet divergence so that substitution into (6.10) leaves it with an overall momentum scale Λ^{-2}. As $G^{(2)}(0, t, g, \Lambda)$ has its leading divergence as t^{-1}, then when (5.10) is substituted into (5.7) the second term can be seen to be $\mathrm{O}(t/\Lambda^2)$ with respect to the first, and so may be dropped in the critical regime. This allows $G^{(2)}(0, t, g, \Lambda)$ to be pulled out as a common factor in (6.4), giving the following expression for the inverse susceptibility:

$$\chi^{-1} = \Gamma^{(2)}(0; t; g, \Lambda)[1 + A_0 g + \delta g^2 + \mathrm{O}(g^3)], \qquad \delta = \theta - \tfrac{1}{4}A_0^2. \tag{6.13}$$

The effect of nonlinear coupling of the field $\phi(x)$ to the external field $H(x)$ has introduced a correction (in the form of a power series in g) to the normal practice of identifying the inverse susceptibility with the two-point vertex function.

7. Method of characteristics

We now seek to find the crossover scaling form for the susceptibility as its critical behaviour changes from being described by the Gaussian fixed point to that described by the Heisenberg fixed point as criticality is approached. This will enable us to derive the form of the susceptibility in the asymptotic regime, and will yield an ε expansion for the amplitude of the leading corrections to scaling. The method is that of Bruce and Wallace (1976). It uses the method of characteristics to relate the vertex functions close to criticality to those far from criticality where a perturbation expansion for the vertex function may be trusted.

We start from the homogeneous RG equation for the vertex functions (3.2) with the RG functions $\beta(g)$, $\gamma_{\phi^2}(g)$, $\gamma_\phi(g)$ having the expansions given in (5.1)–(5.3).

We define a running coupling constant $g(\tau)$ and a running temperature $t(\tau)$ to examine the vertex functions when the cut-off Λ is rescaled by a factor e^τ. Then running quantities are chosen to satisfy

$$dg(\tau)/d\tau = \beta(g(\tau)), \tag{7.1}$$

$$dt(\tau)/d\tau = \gamma_{\phi^2}(g(\tau))t(\tau), \tag{7.2}$$

with the initial conditions

$$g(\tau=0)=g, \qquad t(\tau=0)=t. \tag{7.3}$$

This allows us to solve formally the RG equation (3.2) using the functions defined in (7.1) and (7.2):

$$\Gamma^{(N)}(q;t,g;\Lambda)=\exp\left(-\tfrac{1}{2}N\int_0^\tau \gamma_\phi[g(\tau')]\,d\tau'\right)\Gamma^{(N)}(q;t(\tau),g(\tau),\Lambda\,e^\tau). \tag{7.4}$$

We solve (7.1) and (7.2) subject to the initial conditions (7.3) perturbatively as a power series in g and ε correct to the required order using a form suitable for expanding about the fixed point g^*. The results are

$$\left(\frac{g^*-g(\tau)}{g^*-g}\right)^{\varepsilon/\omega}\left(\frac{g}{g(\tau)}\right)=e^{\varepsilon\tau} \tag{7.5}$$

where

$$\omega=\beta'(g^*)=\varepsilon-(\beta_2/\beta_0^2)\varepsilon^2+O(\varepsilon^3); \tag{7.6}$$

$$t(\tau)=t\left(\frac{g^*-g(\tau)}{g^*-g}\right)^\rho\{1+c[g(\tau)-g]\} \tag{7.7}$$

where

$$\rho=\frac{\gamma_0}{\beta_0}+\varepsilon\left(\frac{\gamma_2}{\beta_0^2}+\frac{2\gamma_0\beta_2}{\beta_0^3}+\frac{\gamma_1}{\beta_0}+\frac{\gamma_0\beta_1}{\beta_0^2}\right), \tag{7.8}$$

$$c=\gamma_0\beta_2/\beta_0^2+\gamma_2/\beta_0; \tag{7.9}$$

$$\exp\left(-\int_0^\tau \gamma_\phi[g(\tau')]\,d\tau'\right)=\left(\frac{g^*-g(\tau)}{g^*-g}\right)^a\{1+b[g(\tau)-g]\} \tag{7.10}$$

where

$$a = -(\gamma_3/\beta_0^2)\varepsilon, \tag{7.11}$$

$$b = -\gamma_3/\beta_0. \tag{7.12}$$

It should be noted that ω, ρ and a are universal.

Using equation (7.4) we may now express $\Gamma^{(2)}(0; g, t, \Lambda)$ in terms of $\Gamma^{(2)}(0; t(\tau), g(\tau), \Lambda\, e^{\tau})$. We simplify by choosing $\Lambda = 1$, and expressing the vertex function in graphical terms using the coefficients defined in (5.9) we obtain

$$\Gamma^{(2)}(0; t, g, 1) = \exp\left(-\int_0^{\tau} \gamma_\phi[g(\tau')]\, d\tau'\right) t(\tau)$$
$$\times\{1 - \tfrac{1}{2} g(\tau) B_1 \ln[t(\tau) e^{-2\tau}] + \tfrac{1}{2} C_1 - \tfrac{1}{2} A_0 + O(\varepsilon g(\tau), g^2(\tau))\}. \tag{7.13}$$

We make a choice of τ such that the perturbation series in $g(\tau)$ may be trusted. A suitable choice is

$$t(\tau) = e^{2\tau} \tag{7.14}$$

which eliminates all powers of logarithms occurring in (7.13). Although the choice (7.14) for τ means that terms which we have neglected as being of order $(t(\tau)/\Lambda^2 e^{2\tau})$ are O(1), this causes no problems as the RG equation (3.2) holds for the leading divergent terms alone, regardless of the presence of less divergent terms that may be of the same order for particular choices of τ (see Bruce and Wallace 1976).

We substitute for $t(\tau)$ and the exponential factor in (7.13) using (7.7) and (7.10) and finally use (7.14) in (7.13) to give the following expression for the inverse susceptibility:

$$\chi^{-1} = t(1 + A_0 g + \delta g^2 + O(g^3))\left(\frac{g^* - g(\tau)}{g^* - g}\right)^{a+\rho}[1 + (c+b)(g(\tau) - g) + O([g(\tau) - g]^2)]$$
$$\times[1 + \tfrac{1}{2}(C_1 - A_0) g(\tau) + O(\varepsilon g(\tau), g^2(\tau))]. \tag{7.15}$$

We may rewrite this expression, separating off all dependence on g into a non-universal overall scale factor $Z(g)$ which is a power series in g correct to the order kept in (7.15):

$$\chi^{-1} = Z(g) t \left(\frac{g^* - g(\tau)}{g^* - g}\right)^{a+\rho}[1 + (c + b + \tfrac{1}{2} C_1 - \tfrac{1}{2} A_0) g(\tau)]. \tag{7.16}$$

Condition (7.14) substituted in (7.5) gives

$$\left(1 - \frac{g(\tau)}{g^*}\right)^{-\varepsilon/\omega + \varepsilon\rho/2} \frac{g(\tau)}{g^*} = t^{-\varepsilon/2}\left(\frac{g}{g^*}\right)\left(1 - \frac{g}{g^*}\right)^{-\varepsilon/\omega + \varepsilon\rho/2} \tag{7.17}$$

Solving (7.17) for g close to g^* yields

$$\frac{g(\tau)}{g^*} = 1 + t^{\bar\omega}[1 - (g^*/g)](g/g^*)^{(1 - 2\bar\omega/\varepsilon)} + O(t^{2\bar\omega}) \tag{7.18}$$

where corrections to scaling are governed by the exponent $\bar\omega$:

$$\bar\omega = \omega\nu = \tfrac{1}{2}\varepsilon - \tfrac{25}{108}\varepsilon^2 + O(\varepsilon^3).$$

Substitution of (7.18) into (7.16) yields an expression explicitly displaying corrections to scaling, with a different overall scale factor $Z'(g)$:

$$\chi^{-1}=Z'(g)t^{\gamma}[1+t^{\bar{\omega}}A(g)], \qquad \gamma=1+\tfrac{1}{6}\varepsilon+\tfrac{25}{324}\varepsilon^2+O(\varepsilon^3), \tag{7.19}$$

$$A(g)=\left\{\frac{1}{3}+\varepsilon\left[\frac{25}{162}+\frac{1}{2\beta_0}(C_1-A_0)+\frac{\gamma_0\beta_2}{\beta_0^3}+\frac{\gamma_2}{\beta_0^2}-\frac{\gamma_3}{\beta_0^2}+\frac{25}{162}\ln\left(\frac{g}{g^*}\right)\right]\right\}\left(1-\frac{g^*}{g}\right). \tag{7.20}$$

Equation (7.20) for the amplitude of the leading correction to scaling is only valid for g close to g^*. For the situation further from criticality the full crossover form (7.16) should be used for the susceptibility.

8. Results and discussion

The model examined in this paper has interesting features as a field theory. The presence of higher-order interactions, normally regarded merely as irrelevant, has of course noticeable effects on non-universal quantities such as the fixed point (5.4) using the contributions calculated from graphs in (5.13). The term in A_0 would not be present in β_1 in the usual ϕ^4 theory. These higher-order interactions also influence the coefficients appearing in the amplitude of corrections to scaling in (7.20). In contrast, universal quantities are found as expected to be independent of the presence of these interactions with explicit cancellations occurring in the graphs contributing to β_2 in (4.5) and in $\gamma_{\phi^2}(g^*)$ in (5.6). The introduction of an external field coupled nonlinearly to the field $\phi(x)$ was found only to change the amplitude for the susceptibility by a prefactor which is a power series in g in (6.13). The analysis performed in § 6 shows that this power series merely contributes to an overall scale factor at the order in ε and g as the coefficients of the power series are not functions of t or Λ. The presence of the power series prefactor would introduce a contribution to the amplitude at $O(\varepsilon^2)$ when a more careful factorisation of (7.15) to extract the overall scale factor $Z(g)$ in (7.16) would have to be performed.

The fixed points have the values shown in table 1 correct to second order in a presumably asymptotic series in ε. It should be noted that the term of $O(\varepsilon^2)$ is much smaller for the BCC lattice than for the SC lattice. To check the convergence of these series it would be desirable to perform a higher-order calculation. This would be rendered very difficult by the need to calculate graphs with more than one loop where the complicated form of the propagator and the shape of the Brillouin zone prohibit the use of many of the common tricks for evaluating higher-order graphs. We must draw what conclusions we may with the figures available from table 1, and can presumably attach more confidence to numerical results for the BCC lattice than for the SC lattice due to the smaller coefficient of the $O(\varepsilon^2)$ corrections to g^*.

The expression for the fixed point of L corresponding to g^* can be evaluated in two ways. We may identify $L^*=2w/g^*$, having evaluated g^* for given ε, or we may express L^* as a power series in ε and then insert the prescribed value for ε. The difference between the two methods is considerable for the SC lattice but only slight for the BCC lattice. We shall use the first method of evaluating L^* as the specification of L in a given dimension uniquely defines g. Using the fact that $w=1$ for the SC lattice and $w=\frac{1}{2}$ for the BCC lattice, with w defined in (2.13), we obtain the following results in three dimensions:

$$L^*=2w/g^*, \tag{8.1}$$

$$d = 3 \qquad L^*_{\text{SC}} = 1.50, \tag{8.2}$$

$$d = 3 \qquad L^*_{\text{BCC}} = 1.10. \tag{8.3}$$

Before these results can be compared with high-temperature series work, the relationship between the number of copies L and the spin s of an Ising system must be considered. The number of possible spin states is the same with L copies and a spin of magnitude $L/2$. The copy model will have a binomial distribution of spin values as opposed to the flat distribution for the spin case. We may therefore identify the L copy model with a spin model of $s = L/2$, but this spin model will be impure with traces of behaviour of lower values of s in addition. This identification is exact for $L = 1$ and, presumably, reasonable for low L. Results (8.2) and (8.3) suggest that the amplitude of confluent singularities should vanish for a spin value slightly greater than $s = \frac{1}{2}$ for both lattice types. There is little in the literature with which to compare this result, but the fact that confluent singularities have proved very difficult to find for the spin-$\frac{1}{2}$ SC model and the spin-$\frac{1}{2}$ BCC model may partially substantiate this result.

Nickel (1981) has shown that the amplitude of the confluent singularities is dependent on the spin s of the BCC Ising model in three dimensions. Further, the amplitude of corrections to scaling does indeed change sign between $s = \frac{1}{2}$ and $s = \infty$, indicating that there is a zero in the amplitude for some s between $s = \frac{1}{2}$ and $s = \infty$.

We may now evaluate expression (7.20) for specific lattice types and a given dimension using the coefficients given in table 1. The general forms of the amplitudes for the SC and BCC lattices are

$$A(g)_{\text{SC}} = \{\tfrac{1}{3} + \varepsilon[0.329 - \tfrac{25}{162}\ln(g/g^*)]\}(1 - g/g^*), \tag{8.4}$$

$$A(g)_{\text{BCC}} = \{\tfrac{1}{3} + \varepsilon[0.334 - \tfrac{25}{162}\ln(g/g^*)]\}(1 - g/g^*). \tag{8.5}$$

With the connection made between spin s and number of copies L (and hence g), the functional form of the amplitude given by (8.5) is seen to be in good agreement with Nickel's findings of the change of sign in the amplitude between $s = \frac{1}{2}$ and $s = \infty$ with a zero in the intermediate region.

Due to the equivalence of $L = 1$ to $s = \frac{1}{2}$ we may use (8.4) and (8.5) to determine the amplitude of corrections to scaling for the spin-$\frac{1}{2}$ Ising model in three dimensions for both the BCC lattice and the SC lattice:

$$A(s = \tfrac{1}{2}, d = 3)_{\text{SC}} = 0.243, \tag{8.6}$$

$$A(s = \tfrac{1}{2}, d = 3)_{\text{BCC}} = 0.062. \tag{8.7}$$

Nickel (1982) estimates the amplitude for the spin-$\frac{1}{2}$ case on the BCC lattice using high-temperature series methods. He does this by evaluating the difference in amplitudes between $s = \frac{1}{2}$ and $s = \infty$ and assigning most of the difference to the amplitude for $s = \frac{1}{2}$. The value he quotes is

$$A(s = \tfrac{1}{2}, d = 3)_{\text{BCC}} = 0.13. \tag{8.8}$$

This value depends crucially on what value for the correction to scaling exponent $\bar{\omega}$ he takes. This number is taken from the RG estimates, but the series in ε which determines $\bar{\omega}$ is poorly convergent and it is only by resumming the series taking into consideration the dominant terms at each order in ε that the value $\bar{\omega} \approx 0.5$ is achieved.

Roskies (1981) also estimates the amplitude of corrections to scaling for the spin-$\frac{1}{2}$ Ising model on the BCC lattice using high-temperature series methods. He directly estimates $A(s=\frac{1}{2}, d=3)_{\mathrm{BCC}}$, but again this value is crucially dependent on the assumption $\bar{\omega}=0.5$. His estimate is

$$0.05 < A(s=\tfrac{1}{2}, d=3) < 0.07. \tag{8.9}$$

While there is very encouraging agreement between (8.9) and (8.7) this must not be relied on too heavily. In determining the amplitude of the leading corrections to scaling as a series in ε in (8.4) and (8.5), it must be noted that the term of $O(\varepsilon)$ is of the same magnitude as the term O(1). In addition, contributions to the amplitude contain terms in the series that determines $\bar{\omega}$ (which is poorly converged as already stated), as can be seen from the examination of (7.16) in conjunction with (7.18). The small magnitude of the amplitude is largely attributed to the proximity of L^* to $L=1$, to which may be attached more confidence due to the size of the $O(\varepsilon^2)$ term for g^*.

The larger magnitude of the amplitude of corrections to scaling in (8.6) for the SC case originates in the fact that L^* is not as close to $L=1$ as it is for the BCC case. The series for g^* appears poorly converged and so no great confidence can be placed in the estimate (8.6). The consistent failure of high-temperature series work (e.g. Gaunt and Sykes 1979) to detect confluent singularity terms may indicate that the evaluation of L^* to higher orders would push L^* closer to $L=1$, so reducing the amplitude given by (8.4).

In summary, the analysis carried out in this paper bears out the qualitative features of high-temperature series work on amplitudes of corrections to scaling (Nickel 1981), and in addition suggests a good numerical comparison with high-temperature series estimates for amplitudes (Nickel 1982, Roskies 1981).

Acknowledgments

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Appendix 1. Simple cubic lattice

We consider the evaluation of the integrals $I_1(t, \Lambda)$ and $I_2(t, \Lambda)$ defined in (5.7) and (5.8) for a SC lattice with lattice spacing Λ^{-1}. The sum over lattice vectors in (3.6) may be performed with the result

$$\hat{G}_0^{(2)}(q, t, \Lambda) = \frac{2\sum_{i=1}^{d}\cos(q_i/\Lambda)}{2d\Lambda^2 K_c/K - 2\Lambda^2\sum_{i=1}^{d}\cos(q_i/\Lambda)} \tag{A1.1}$$

with $\boldsymbol{q}$ a d-dimensional vector with components q_i, $i=1,\ldots,d$, and the coordination number of the lattice $\nu=2d$. Integrals over the Brillouin zone of the lattice take the following simple form for any function $F(q)$:

$$\int^{\mathrm{BZ}} \mathrm{d}^d q\, F(q) = \prod_{i=1}^{d}\left(\int_{-\pi\Lambda}^{\pi\Lambda}\mathrm{d}q_i\right)F(q). \tag{A1.2}$$

The combinations of integrals that must be evaluated are given in (5.11) and (5.12) for $\Sigma_1(0, t)$ and $\Pi_1(0, t)$.

We shall give a detailed analysis of $I_2(t, \Lambda)$; corresponding analyses can be carried out for the other integrals appearing in (5.11) and (5.12):

$$I_2(t,\Lambda)=\frac{1}{(2\pi)^d}\prod_{i=1}^{d}\left[\left(\int_{-\pi\Lambda}^{\pi\Lambda}\mathrm{d}q_i\right)\left(\frac{2\,\Sigma_{i=1}^{d}\cos(q_i/\Lambda)}{2d\Lambda^2K_c/K-2\Lambda^2\,\Sigma_{i=1}^{d}\cos(q_i/\Lambda)}\right)^2\right]. \tag{A1.3}$$

We rescale the momenta $q_i'=q_i\Lambda^{-1}$ to make explicit the momentum scale of the integral. This factor of $\Lambda^{-\varepsilon}$ will be cancelled by Λ^{ε} coming from the coupling constant, and it will be omitted in the following as will the primes on the rescaled momenta:

$$\begin{aligned}I_2(t,\Lambda)&=\frac{1}{(2\pi)^d}\prod_{i=1}^{d}\left(\int_{-\pi}^{\pi}\mathrm{d}q_i\right)\left(1-\frac{2(2dK_c/K)}{(2dK_c/K-2\,\Sigma_{i=1}^{d}\cos q_i)}\right.\\&\qquad\left.+\frac{(2dK_c/K)^2}{(2dK_c/K-2\,\Sigma_{i=1}^{d}\cos q_i)^2}\right)\\&=1+(2\pi)^{-d}\prod_{i=1}^{d}\left(\int_{-\pi}^{\pi}\mathrm{d}q_i\right)(z^2/y^2-2z/y)\end{aligned} \tag{A1.4}$$

where

$$z=2dK_c/K=2d(1+t\Lambda^{-2}), \tag{A1.5}$$

$$y=z-2\sum_{i=1}^{d}\cos q_i>0 \qquad \text{for } t>0. \tag{A1.6}$$

We now express the integrand in (A1.4) as a Laplace transform with respect to the variable y. This transform is well defined for $y>0$

$$I_2(t,\Lambda)=1+(2\pi)^{-d}\prod_{i=1}^{d}\left(\int_{-\pi}^{\pi}\mathrm{d}q_i\right)\left(\int_0^{\infty}\mathrm{d}s\;\mathrm{e}^{-sy}(z^2s-2z)\right). \tag{A1.7}$$

We now change the order of integration in (A1.7) and use

$$\mathrm{e}^{-sy}=\exp\left[-s\left(z-2\sum_{i=1}^{d}\cos q_i\right)\right]=\mathrm{e}^{-sz}\prod_{i=1}^{d}\exp(2s\cos q_i) \tag{A1.8}$$

and the identity

$$\mathrm{e}^{-sz}=\mathrm{e}^{-2ds}\exp(-2dst\Lambda^{-2}) \tag{A1.9}$$

to factorise the momentum integrals, yielding

$$I_2(t,\Lambda)=1+\int_0^{\infty}\mathrm{d}s\left[\exp(-2dst\Lambda^{-2})(z^2s-2z)\left(\frac{\mathrm{e}^{-2s}}{2\pi}\int_{-\pi}^{\pi}\exp(2s\cos q)\,\mathrm{d}q\right)^d\right]. \tag{A1.10}$$

The one-dimensional momentum integral may be performed to give a modified Bessel function of order zero which allows (A1.10) to be expanded as a power series in $\varepsilon=4-d$:

$$\begin{aligned}I_2(t,\Lambda)=1&+\int_0^{\infty}[\exp(-8st\Lambda^{-2})(64s-16)(I_0(2s)\,\mathrm{e}^{-2s})^4]\,\mathrm{d}s-\varepsilon\int_0^{\infty}\{\exp(-8st\Lambda^{-2})\\&\times(I_0(2s)\,\mathrm{e}^{-2s})^4[(64s-16)\ln[I_0(2s)\,\mathrm{e}^{-2s}]+32s-4]\}\,\mathrm{d}s\\&+\mathrm{O}(\varepsilon^2)+\mathrm{O}(t/\Lambda^2).\end{aligned} \tag{A1.11}$$

From the first and second terms in (A1.11) the coefficients B_2 and C_2 defined in (4.10) can be obtained and from the third term E_2 and D_2 can be obtained. Similar methods can be used to determine the coefficients in (5.9). The results are noted in table 1.

Appendix 2. Body centred cubic lattice

For a BCC lattice with lattice spacing Λ^{-1} in d dimensions then the sum over lattice vectors in (3.6) may be performed yielding

$$\hat{G}_0^{(2)}(q,t,\Lambda)=\frac{2^d\,\Pi_{i=1}^d\cos(q_i/2\Lambda)}{2^d\Lambda^2K_c/K-2^d\,\Pi_{i=1}^d\cos(q_i/2\Lambda)} \tag{A2.1}$$

with $\boldsymbol{q}$ a d-dimensional vector with components q_i, $i=1,\ldots,d$, and the coordination number $\nu=2^d$.

Integrals over the Brillouin zone of a BCC lattice in d dimensions have a very complex form. We may use the general result which holds for any general dimension for integrands which are of a certain functional form,

$$\frac{1}{(2\pi)^d}\int^{\mathrm{BZ}}\mathrm{d}^dq\,F\bigg(\prod_{i=1}^d\cos\frac{q_i}{2\Lambda}\bigg)$$
$$=\frac{1}{(2\pi)^d}\prod_{i=1}^d\bigg(\int_{-\pi\Lambda}^{+\pi\Lambda}\mathrm{d}^dq_i\bigg)\bigg[F\bigg(\prod_{i=1}^d\cos\frac{q_i}{2\Lambda}\bigg)+F\bigg(-\prod_{i=1}^d\cos\frac{q_i}{\Lambda}\bigg)\bigg] \tag{A2.2}$$

which can be proved using the symmetry properties of the Brillouin zone. This allows us to proceed to evaluate the combinations of integrals given in (5.11) and (5.12) for $\Sigma_1(0,t)$ and $\Pi_1(0,t)$.

We shall give a detailed analysis for $I_2(t,\Lambda)$; corresponding analyses can be carried out for the other integrals appearing in (5.11) and (5.12).

Using (A2.1) and (A2.2), we may recast (5.8) in the form

$$I_2(t,\Lambda)=\frac{1}{(2\pi)^d}\prod_{i=1}^d\bigg(\int_{-\pi\Lambda}^{\pi\Lambda}\mathrm{d}q_i\bigg)\bigg[\bigg(\frac{2^d\,\Pi_{i=1}^d\cos(q_i/2\Lambda)}{2^d\Lambda^2K_c/K-2^d\,\Pi_{i=1}^d\cos(q_i/\Lambda)}\bigg)^2$$
$$+\bigg(\frac{2^d\,\Pi_{i=1}^d\cos(q_i/2\Lambda)}{2^d\Lambda^2K_c/K+2^d\,\Pi_{i=1}^d\cos(q_i/2\Lambda)}\bigg)^2\bigg]. \tag{A2.3}$$

We rescale the momenta $q_i'=q_i/(2\Lambda)$ to make explicit the momentum scale of the integral. This factor of $\Lambda^{-\varepsilon}$ will be cancelled by Λ^{ε} coming from the coupling constant, and it will be omitted in the following, as will the primes on the rescale momenta. We start from the following expansion and change of variables:

$$I_2(t,\Lambda)=\frac{1}{\pi^d}\prod_{i=1}^d\bigg(\int_{-\pi/2}^{\pi/2}\mathrm{d}q_i\bigg)\bigg\{2-2\bigg(\frac{K/K_c}{K/K_c-\Pi_{i=1}^d\cos q_i}+\frac{K/K_c}{K/K_c+\Pi_{i=1}^d\cos q_i}\bigg)$$
$$+\bigg[\bigg(\frac{K/K_c}{K/K_c-\Pi_{i=1}^d\cos q_i}\bigg)^2+\bigg(\frac{K/K_c}{K/K_c+\Pi_{i=1}^d\cos q_i}\bigg)^2\bigg]\bigg\}$$
$$=2+\frac{1}{\pi^d}\prod_{i=1}^d\bigg(\int_{-\pi/2}^{\pi/2}\mathrm{d}q_i\bigg)\bigg(-\frac{2}{1+\alpha y}-\frac{2}{1-\alpha y}+\frac{1}{(1+\alpha y)^2}+\frac{1}{(1-\alpha y)^2}\bigg) \tag{A2.4}$$

where

$$y = \prod_{i=1}^{d} \cos q_i \leq 1, \tag{A2.5}$$

$$\alpha = K_c/K = (1 + t\Lambda^{-2})^{-1} < 1 \qquad \text{for } t > 0. \tag{A2.6}$$

We now express the integral of (A2.4) as an inverse Mellin transform with respect to the variable y defined in (A2.5). This is well defined for the condition $\alpha < 1$ in (A2.6):

$$I_2(t, \Lambda) = 2 + \frac{1}{\pi^d} \prod_{i=1}^{d} \left(\left(\int_{-\pi/2}^{\pi/2} \mathrm{d}q_i \right) \left(\frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \frac{\mathrm{d}s}{s} \exp(-s \ln y) [F(2, s, s+1; \alpha) + F(2, s, s+1; -\alpha) - 2F(1, s, s+1; \alpha) - 2F(1, s, s+1; -\alpha)] \right) \tag{A2.7}$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric function, s is a complex variable, and $0 < C < 1$ for the integrals to be well defined. We interchange the order of integration where the choice of transform enables the integrals over the momentum components to be factorised, yielding

$$I_2(t, \Lambda) = 2 + \frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \frac{\mathrm{d}s}{s} [F(2, s, s+1; \alpha) + F(2, s, s+1; -\alpha) - 2F(1, s, s+1; \alpha) - 2F(1, s, s+1; -\alpha)] \times \left[\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \mathrm{d}q \exp(-s \ln \cos q) \right]^d. \tag{A2.8}$$

Using the result

$$\frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} \mathrm{d}q \exp(-k \ln \cos q) = [\Gamma(\tfrac{1}{2} - \tfrac{1}{2}k)/\Gamma(1 - \tfrac{1}{2}k)\pi^{1/2}] \qquad \text{for } k < 1 \tag{A2.9}$$

equation (A2.8) takes the final form

$$I_2(t, \Lambda) = 2 + \frac{1}{2\pi \mathrm{i}} \int_{C-\mathrm{i}\infty}^{C+\mathrm{i}\infty} \frac{\mathrm{d}s}{s} [F(2, s, s+1; \alpha) + F(2, s, s+1; -\alpha) - 2F(1, s, s+1; -\alpha) - 2F(1, s, s+1; \alpha)][\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s)/\Gamma(1 - \tfrac{1}{2}s)\pi^{1/2}]^d. \tag{A2.10}$$

Equation (A2.10) may be evaluated by contour integration techniques. The integral may be closed at infinity in the negative half plane, with the result that $I_2(t, \Lambda)$ is given by the sum of the residues of the hypergeometric functions which have poles for s equal to a negative integer.

For a general term in (A2.10) we have, for an integer n,

$$\lim_{s \to -n, n \neq 0} F(\nu, s, s+1, \alpha) s^{-1} = \Gamma(-n)(\nu)_n (-1)^n \alpha^n F(\nu + n, 0, n+1, \alpha) \tag{A2.11}$$

where

$$F(\nu + n, 0, n+1, \alpha) \equiv 1, \qquad (\nu)_n = \Gamma(\nu + n)/\Gamma(\nu).$$

The residue from the hypergeometric function is proportional to the residue of the gamma function and, together with the result that for $s=0$ the residue is equal to -2, we have for (A2.10)

$$I_2(t,\Lambda)=2-2+\sum_{n=1}^{\infty}\left(\frac{\alpha^n+\alpha^{-n}}{n!}\right)\left[\frac{\Gamma(\frac{1}{2}+\frac{1}{2}n)}{\Gamma(1+\frac{1}{2}n)\pi^{1/2}}\right]^d[(2)_n-2(1)_n]$$

$$=\sum_{m=1}^{\infty}\alpha^{2m}(4m-2)[\Gamma(\tfrac{1}{2}+m)/\Gamma(1+m)\pi^{1/2}]^d. \tag{A2.12}$$

Expanding (A2.12) as a power series in ε yields

$$I_2(t,\Lambda)=\pi^{1/2}\sum_{m=1}^{\infty}\alpha^{2m}(4m-2)\frac{\Gamma^4(\frac{1}{2}+m)}{\Gamma^4(1+m)}-\frac{\varepsilon}{\pi^2}\times\sum_{m=1}^{\infty}\alpha^{2m}(4m-2)\frac{\Gamma^4(\frac{1}{2}+m)}{\Gamma^4(1+m)}\ln\left(\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(1+m)\pi^{1/2}}\right)+O(\varepsilon^2). \tag{A2.13}$$

From (A2.13) the coefficients B_2, C_2, D_2, E_2 and F_2 defined in (5.10) can be found and are listed in table 1.

Appendix 3. The n-component spin model

We consider the generalisation of the model to the case where the spin variable, $\boldsymbol{s}_i$, has n components. The O(n) symmetric reduced Hamiltonian is given by, with the symmetry breaking external field $\boldsymbol{H}_i$,

$$-\frac{\mathcal{H}}{kT}=\tfrac{1}{2}\sum_{\langle ij\rangle}K_{ij}\boldsymbol{s}_i\cdot\boldsymbol{s}_j-\sum_i\boldsymbol{H}_i\cdot\boldsymbol{s}_i. \tag{A3.1}$$

We take the spin at site i, $\boldsymbol{s}_i$, to be the sum of L copies of classical n-component spins $\boldsymbol{\sigma}_{im}$, $m=1,2,\ldots,L$ normalised by a factor of $L^{-1/2}$

$$\boldsymbol{s}_i=L^{-1/2}\sum_{m=1}^{L}\boldsymbol{\sigma}_{im},\qquad |\boldsymbol{\sigma}_{im}|=n^{1/2}. \tag{A3.2}$$

The configuration sum over the $\boldsymbol{s}_i$ in evaluating the partition function Z becomes the product of L integrals over the solid angle of an n-dimensional hypersphere at each site of the lattice;

$$Z=\sum_{\{s\}}\exp(-\mathcal{H}/kT)=\prod_{i=1}^{N}\prod_{m=1}^{L}\int d\Omega_{im}\exp(-\mathcal{H}/kT)$$

where the reduced Hamiltonian is expressed in terms of the variables $\boldsymbol{\sigma}_{im}$.

Performing a Hubbard transformation in the variable $\boldsymbol{\phi}_i$ with n components to extract a field theory yields

$$Z=\prod_{i=1}^{N}\prod_{i=1}^{L}\int d\Omega_{im}\int D\phi\,\exp\left(-\tfrac{1}{2}\sum_{ij}K_{ij}^{-1}\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j+\sum_i(\boldsymbol{\phi}_i+\boldsymbol{H}_i)\cdot\sum_{m=1}^{L}L^{-1/2}\boldsymbol{\sigma}_{im}\right). \tag{A3.3}$$

The integration over solid angles in (A3.3) reduces to performing

$$Z=\int D\phi\,\exp\Big(-\tfrac{1}{2}\sum_{ij}K_{ij}^{-1}\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j\Big)\Big[\prod_{i=1}^{N}\prod_{m=1}^{L}\Big(s_{n-1}\int_0^{\pi}\mathrm{d}\theta_{im}\,\sin^{n-2}\theta_{im}\times\exp\{[(\phi_i+H_i)(Ln)^{1/2}\cos\theta_{im}]\}\Big)\Big] \tag{A3.4}$$

where s_{n-1} is the surface area of an $(n-1)$-dimensional sphere and (ϕ_i+H_i) is the modulus of $(\boldsymbol{\phi}_i+\boldsymbol{H}_i)$. The one-dimensional integral in (A3.4) may be performed with the result

$$Z=\int D\phi\,\exp\Big\{-\tfrac{1}{2}\sum_{ij}K_{ij}^{-1}\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j+\sum_i L\ln\Big[s_{n-1}\Big(\frac{2n^{1/2}L^{1/2}}{(\phi_i+H_i)}\Big)^{n/2-1}\times\pi^{1/2}\Gamma\Big(\frac{n}{2}-\frac{1}{2}\Big)I_{n/2}\Big(\frac{(\phi_i+H_i)}{n^{1/2}L^{1/2}}\Big)\Big]\Big\} \tag{A3.5}$$

where $I_{n/2-1}(x)$ is a Bessel function.

The interaction terms may again be obtained by expanding the Bessel function as a power series in $\boldsymbol{\phi}_i$ for zero external field. Defining the dimensionless coupling constant g, with w defined in (2.13), by

$$g=4!w/4(n+2)L \tag{A3.6}$$

we may decompose the reduced Hamiltonian into a quadratic term and interaction terms ordered as a power series in g. On taking the thermodynamic limit and using a rescaled field variable in momentum space $\phi(q)$ defined in (2.15), we obtain for these two terms

$$\mathcal{H}^{(2)}=-(2\pi)^{-d}\int^{\mathrm{BZ}}\mathrm{d}^dq\,\tfrac{1}{2}(\Lambda^2K^{-1}(q)-\Lambda^2)\boldsymbol{\phi}(q)\cdot\boldsymbol{\phi}(-q), \tag{A3.7}$$

$$\mathcal{H}_{\mathrm{int}}=-\frac{g\Lambda^{\varepsilon}}{4!}\frac{1}{(2\pi)^{3d}}\prod_{m=1}^{3}\Big(\int^{\mathrm{BZ}}\mathrm{d}^dq_m\,\boldsymbol{\phi}(q_m)\Big)\boldsymbol{\phi}\Big(-\sum_{m=1}^{3}q_m\Big)+g^2\frac{20(n+2)}{3(n+4)}\frac{\Lambda^{2\varepsilon-2}}{6!}\frac{1}{(2\pi)^{5d}}\prod_{m=1}^{5}\Big(\int\mathrm{d}^dq_m\,\boldsymbol{\phi}(q_m)\Big)\boldsymbol{\phi}\Big(-\sum_{m=1}^{5}q_m\Big)+\mathrm{O}(g^3). \tag{A3.8}$$

In (A3.8) scalar products between the n-component fields must be taken in order to maintain the $\mathrm{O}(n)$ symmetry of the Hamiltonian.

The theory of this n-component model now parallels the discussion of the one-component model using the usual techniques for handling n-component fields. There are two sources of dependence on the number of components; that arising from the combinatoric factors of graphs and that coming explicitly from the coupling constants in (A3.8). In terms of the coefficients defined in (5.9) and (5.10) the expression for the fixed point g^* reads

$$g^*=\frac{6}{(n+8)B_2}\Big\{1+\varepsilon\Big[\frac{3(3n+14)}{(n+8)^2}+B_2^{-1}\Big(C_2-D_2+\frac{4(n+2)}{(n+8)}A_0\Big)\Big]\Big\}. \tag{A3.9}$$

Using the prescription

$$L^*=4!W/4(n+2)g^* \tag{A3.10}$$

we obtain the following results for the SC and BCC lattices in three dimensions (using table 1 for B_2, C_2 and A_0):

$$L^*_{\rm SC}(n=2)=1.216, \tag{A3.11}$$

$$L^*_{\rm SC}(n=3)=1.048, \tag{A3.12}$$

$$L^*_{\rm BCC}(n=2)=0.880, \tag{A3.13}$$

$$L^*_{\rm BCC}(n=3)=0.752. \tag{A3.14}$$

Again the expression for g^* as an ε expansion appears to be much better convergent for the BCC case than the SC case and so more reliance may be placed on figures obtained for the BCC lattice.

The analysis of § 6 for the susceptibility gives the same qualitative results as before, that is modifying the relation between the susceptibility and the two-point vertex function by a power series in g with different numerical coefficients. This allows the analysis of § 7 to proceed as before to obtain the general form of the amplitude of the leading corrections to scaling as

$$A(g)=\left\{K_1+\varepsilon\left[K_2+\frac{\gamma_0\beta_2}{\beta_0^3}+\frac{\gamma_2}{\beta_0^2}-\frac{\gamma_3}{\beta_0^2}+K_1K_3\ln\left(\frac{g}{g^*}\right)\right]\right\}\left(1-\frac{g^*}{g}\right) \tag{A3.15}$$

with

$$K_1=\frac{n+2}{n+8}, \qquad K_2=\frac{(n+2)(n^2+22n+52)}{2(n+8)^3},$$

$$K_3=\frac{(-n^2+8n+68)}{2(n+8)^2}, \qquad \frac{\gamma_3}{\beta_0^2}=\frac{(n+2)}{2(n+8)^2},$$

$$\frac{\gamma_0}{\beta_0}=\frac{(n+2)}{(n+8)}, \qquad \frac{\beta_2}{\beta_0^2}=\frac{3(3n+14)}{(n+8)^2}.$$

Using (A3.15) we may predict the amplitude for the leading corrections to scaling for the classical Heisenberg models with $n=2$ and $n=3$ in three dimensions for the two lattice types with the results:

$$A(n=2)_{\rm SC}=0.216, \tag{A3.16}$$

$$A(n=3)_{\rm SC}=0.040, \tag{A3.17}$$

$$A(n=2)_{\rm BCC}=-0.129, \tag{A3.18}$$

$$A(n=3)_{\rm BCC}=-0.359. \tag{A3.19}$$

Information from high-temperature series for the $n=2$ and $n=3$ cases is much more sparse than for the Ising model, and the shorter series do not enable accurate determinations to be made of exponents (see Rushbrooke *et al* 1974 for a review) and no confirmation of the presence of confluent singularities has yet been attempted. Again the estimates (A3.18) and (A3.19) are to be trusted more than (A3.16) and (A3.17) due to the better determination of g^* for the BCC lattice. The most striking result from (A3.18) and (A3.19) is that the amplitude of the confluent singularities should have the opposite sign for $n=2$ and $n=3$ from the Ising model, and this qualitative feature may be easier to determine from high-temperature series work.

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